



THE PAINLEVÉ PARADOXES AND THE DYNAMICS OF A BRAKE SHOE†

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Using experience gained in studies of the celebrated Painlevé–Klein example [1, 2], a mathematical model of a brake shoe is constructed that avoids the Painlevé paradoxes. A qualitative analysis of this model (using, in particular, the method of point mappings) has enabled the nature of the possible motions of a brake shoe to be ascertained, and has enabled self-excited oscillations to be observed which may be attributed to dry friction with a characteristic curve, no part of which is descending. As far as is known, this is the first record of this phenomenon, quite normal for automatic control systems, in simple mechanical systems (without servoconstraints).

The Painlevé paradoxes [3] were encountered [4] when equations of motion were set up for a brake shoe. It turned out that for certain relationships between the physical parameters the equations of motion are either inconsistent or indeterminate. Postulating that a brake shoe, upon contact with a wheel, experiences “frictional impact”, possible consequences of this assumption were considered and the following conclusions ensued: the lack of experimental data precludes proposing any definitive solution of the problem; up to the present there are no conclusive answers to the questions raised.

In this paper answers will be proposed to these questions, in any case for a brake shoe: equations of motion, avoiding the Painlevé paradoxes, will be formulated taking into account either tangential or normal elasticities of the shoe. These equations will be used to make a qualitative study of brake-shoe dynamics and the conditions for impact interactions of the shoe with the wheel to occur when the tangential stiffness increases without limit; self-excited oscillations of the shoe are observed when its normal stiffness is taken into consideration.

1. THE EQUATIONS OF MOTION OF A BRAKE SHOE AND THE PAINLEVÉ PARADOXES

Let P be a brake shoe, acted on by a constant force F , capable of rotating about a fixed vertical axis O_1 (see Fig. 1). A wheel Q , driven by a constant torque M , rotates about a fixed axis O parallel to O_1 . The bodies P and Q interact on contact, through forces of Coulomb (dry) friction.

According to the notation in Fig. 1, the equations of motion of the system, based on the laws of classical mechanics that apply when the shoe is clamped onto the wheel, are

$$J\dot{\omega} = M - rT, \quad b(N - F) = aT, \quad T = -f|N|\operatorname{sgn}\omega \quad (1.1)$$

where f is the coefficient of sliding friction, J is the moment of inertia of the wheel and T and N are the tangential and normal components of the interaction forces of the wheel and the shoe. It follows from (1.1) that

$$N = bF[b - af \operatorname{sgn}(\omega N)]^{-1} \quad (1.2)$$

Assigning different signs to the quantities F , ω , $b - af$ and N and determining the sign of N from (1.2), we obtain a table in which the lowest row lists the signs of the normal reaction N found from (1.2) according to the signs of F , ω , $b - af$ and N postulated in the previous rows.

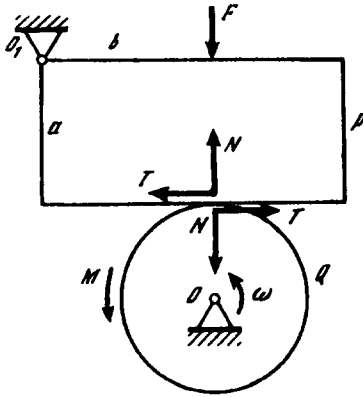


Fig. 1.

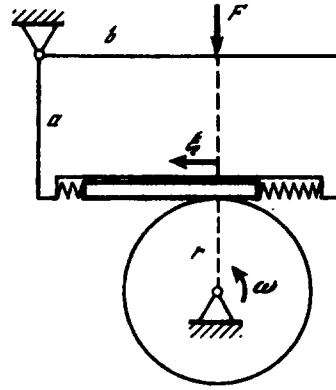


Fig. 2.

F	+	+	+	+	+	+	+	-	-	-	-	-	-	-
ω	+	+	+	+	-	-	-	+	+	+	+	-	-	-
$b-af$	+	+	-	-	+	+	-	+	+	-	-	+	+	-
N	+	-	+	-	+	-	+	+	-	+	-	+	-	+
N	+	+	-	+	+	+	+	-	-	-	+	-	-	+

This table has been drawn up on the assumption that the shoe-wheel constraint is bilateral. In the case of a unilateral constraint the assumption $N < 0$ must be excluded. The Painlevé paradoxes state that, as follows from the table, when $b-af < 0, \omega > 0$, neither of these hypotheses works: if one assumes that $N > 0$, it follows from (1.2) that $N < 0$; but if one assumes that $N < 0$, it follows from (1.2) that $N > 0$. If $\omega < 0$, however, both hypotheses $N > 0$ and $N < 0$ are admissible.

We know [1-3, 5, 6] that the Painlevé paradoxes can be eliminated by taking the longitudinal or transverse elasticity of one of the touching bodies into account. We shall deal here with two models of a brake shoe: in the first, allowance is made for the longitudinal (tangential) stiffness k of the shoe; in the second, the active factor will be the transverse (normal) stiffness k_1 .

2. ALLOWANCE FOR THE TANGENTIAL ELASTICITY OF THE SHOE

Figure 2 is a working model of a braking device allowing for tangential elasticity: a plate Π of negligibly small mass, attached to the shoe by springs, can move along the shoe. Let ξ be the displacement of the plate from the position at which the springs are slack. We set up the equations of motion on the assumption that the plate Π , interacting with the rotating wheel by Coulomb forces, does not slide along the wheel, that is to say, the following relation holds

$$\xi' = r\omega \tag{2.1}$$

By Coulomb's law, a necessary condition for this to occur is

$$k|\xi| \leq f|N| \tag{2.2}$$

In that case the equations of dynamics may be written as

$$J\omega' = M - rk\xi, \quad b(N - F) = ak\xi \tag{2.3}$$

Equations (2.1) and (2.3) describe the motion of the representative point in the phase plane (ξ, ω) in a strip containing the origin, whose boundaries are determined from condition (1.2).

Define dimensionless variables by $\xi_0 = \xi r^{-1}, \tau = trk^{1/2}J^{-1/2}, F_0 = fFk^{-1}r^{-1}, N_0 = fnK^{-1}r^{-1}, M_0 = Mk^{-1}r^{-2}, \sigma = afb^{-1}$. Then, retaining the previous notation for the quantities ξ_0, F_0, N_0, M_0 , we write the equations of motion as

$$\xi' = \omega, \quad \omega' = M - \xi \tag{2.4}$$

Hence it follows that the phase trajectories form a family of concentric circles about the point $\xi = M, \omega = 0$; if allowance is made for the small viscous friction, they form a family of twisted spirals. It follows from (2.2) and (2.3) that the domain Φ in the phase plane (ξ, ω) in which Eqs (2.4) hold is determined by the inequalities $|\xi| \leq |F + \sigma\xi|$, whence it follows that the boundaries of Φ are the numbers ξ_1 and ξ_2 defined by

$$\xi_1 = F(1 - \sigma)^{-1}, \quad \xi_2 = -F(1 + \sigma)^{-1} \tag{2.5}$$

and the domain Φ itself is defined by the inequalities $\xi_2 < \xi < \xi_1$ if $\sigma < 1, F > 0$; $\xi_1 < \xi < \xi_2$ if $\sigma < 1, F < 0$; $\xi > \xi_2$ and $\xi > \xi_1$ if $\sigma > 1, F > 0$; $\xi > \xi_1$ and $\xi < \xi_2$ if $\sigma > 1, F < 0$.

Outside Φ the phase trajectories form a family of straight lines parallel to the ξ axis, along which the representative point moves abruptly up to the boundary of Φ . The physical meaning of these jumps is an infinitely rapid sliding movement of the mass-less plate subject to a finite force.

The nature of the division of the phase plane into trajectories depends on the relations of the physical parameters. In this problem there are three essential physical parameters: F, σ and M , which may take arbitrary values in the following ranges: $-\infty < F < \infty, 0 \leq \sigma < \infty, 0 \leq M < \infty$. The qualitative features of the phase portrait in the (ξ, ω) plane depend on the mutual positions of the points ξ_0, ξ_1, ξ_2 and the origin. It follows from (2.5) and the fact that $\xi_0 = M$ that

$$\begin{aligned} \xi_1 - \xi_0 &= \lambda(1 - \sigma)^{-1}, \quad \xi_1 - \xi_2 = 2F(1 - \sigma)^{-1} \\ \xi_0 - \xi_2 &= (\lambda + 2M)(1 + \sigma)^{-1}, \quad \lambda = F + M(\sigma - 1) \end{aligned} \tag{2.6}$$

On the basis of (2.5) and (2.6), one can state that the bifurcation boundaries on the parameter half-plane $(F, \sigma), (\sigma > 0)$ are the straight lines $F = 0, \sigma = 1, \lambda = 0, \lambda = -2M$, which divide the half-plane into eight domains, as shown in Fig. 3. The phase portraits corresponding to the parameter values in these domains are shown in Fig. 4. The motion of the representative point along the boundary $\xi = \xi_1$ or $\xi = \xi_2$ occurs only in sections where the phase trajectories merge [7, 8] and is governed by the equations

$$\dot{\omega} = \xi_0 - \xi_{1,2} \tag{2.7}$$

The direction in which the representative point moves along the boundary is determined by the sign of the right-hand side of Eqs (2.7), as given by (2.6). The phase portraits in Fig. 4 and the division of the parameter plane into domains corresponding to different types of behaviour (Fig. 3) represent a complete solution of the dynamical problem formulated at the beginning of this paper, on the assumption that the constraint at the shoe-wheel contact is bilateral. According to these phase portraits, the shoe-wheel system, after several "frictional impacts" or otherwise, may approach equilibrium or convert into uniformly accelerated motion of the wheel. In the usual case of a unilateral constraint, the shoe and wheel may only press on one another, and therefore there is an additional condition $N \geq 0$, which reduces to the inequality

$$\xi \geq \xi_* = -F\sigma^{-1}, \tag{2.8}$$

following from the second equation of (2.3).

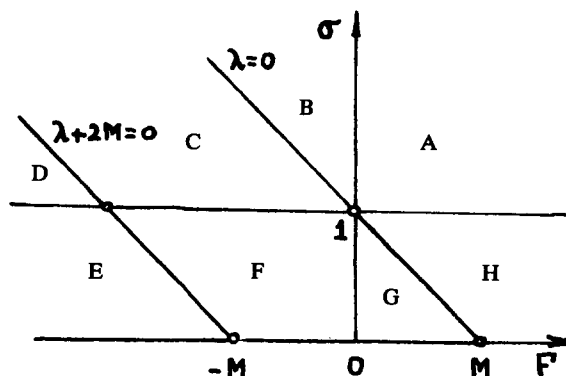


Fig. 3.

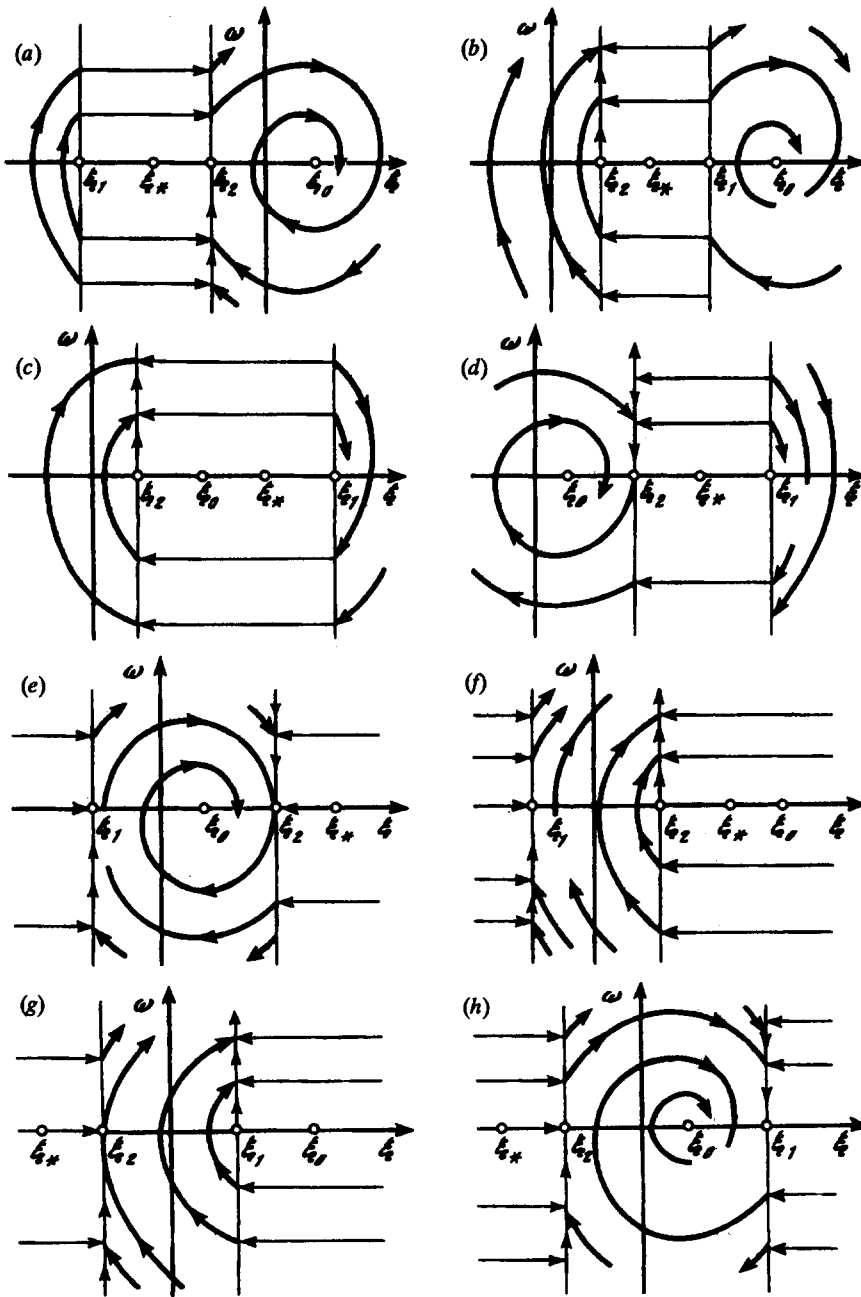


Fig. 4.

In each phase portrait of Fig. 4 we have plotted a point ξ_0 indicating in which part of the plane the shape of the phase portrait is preserved. Beyond that part $N < 0$. It can be seen from the cases presented in Fig. 4 that only in cases A, H and G does the phase point never leave the domain of phase space defined by the condition $\xi > \xi_0$. In all other cases, except case B, it leaves that domain. Case B corresponds to two possibilities: the phase point will remain in the domain $N > 0$, approaching the stable equilibrium position ξ_0 , or the point will leave the domain, finally moving with uniform acceleration along the line $\xi = \xi_2$. The domain of attraction of the stable equilibrium ξ_0 is not large and it becomes smaller as k increases, since $\xi_0 - \xi_1 \rightarrow 0$ as $k \rightarrow \infty$. This case of paradoxical equilibrium has been pointed out before [4], but when $k = \infty$ it must be regarded as unrealizable.

3. ALLOWANCES FOR NORMAL ELASTICITY

When allowance is made for normal elasticity, the working model of the brake shoe-wheel system is that shown in Fig. 5. Unlike the previous model, the mass-less plate attached to the shoe by springs of stiffness k_1 may be displaced only transversely. It is also assumed that some arresting device or stop is placed on the either side of the shoe; the shoe, after hitting the stop, immediately experiences an impact with coefficient of restitution $\kappa < 1$. The angle of rotation ψ of the shoe is measured from the position at which the plate touches the wheel and the springs are slack. The angles of rotation of the wheel and the shoe are considered positive in the anticlockwise sense.

The equations of motion of the shoe-wheel system in the domain $\psi < 0$ (J_1 and J are the moments of inertia of the shoe and the wheel, respectively) are

$$J_1 \psi'' = (af \operatorname{sgn} \omega - b)k_1 b \psi - bF, \quad J \omega' = M - rfbk_1 \operatorname{sgn} \omega \tag{3.1}$$

For $\psi = 0$ and $\psi' > 0$, when the shoe-wheel contact is broken, an impact occurs against the stop, so that the velocity ψ'_+ after impact is related to the pre-impact velocity ψ'_- by

$$\psi'_+ = -\kappa \psi'_- \quad (0 \leq \kappa \leq 1) \tag{3.2}$$

In dimensionless variables $\tau = btk^{1/2}J^{-1/2}$, $\sigma = afb^{-1}$, $F_0 = Fb^{-1}k_1^{-1}$, $M_0 = M(rfbk_1)^{-1}$, $J_0 = bJ(brJ_1)^{-1}$ the equations of motion (2.1) and (2.2) for $\omega > 0$, $\psi \leq 0$ are as follows (omitting the zero subscript from now on)

$$\psi'' = (\sigma - 1)\psi - F, \quad J \omega' = \psi \operatorname{sgn} \psi + M \quad (\psi < 0) \tag{3.3}$$

$$\psi'_+ = \psi'_-, \quad J \omega' = M \quad (\psi = 0) \tag{3.4}$$

Equations (3.1) describe the motion of the representative point in the three-dimensional phase space (ψ, ψ', ω) , $\psi < 0$. It follows from these equations that, for any initial data in the domain $\omega < 0$, the representative point reaches the half-plane $\omega \geq 0$ in a finite span of time and then never leaves it. It is therefore sufficient to study the structure of the phase space in the domain $\omega \geq 0$, i.e. to investigate the solution of Eqs (3.3) and (3.4). On the boundary $\omega = 0$ of the half-space there is a domain $\psi \leq -M$ of "sliding" motions [7] in which the representative point moves subject to the equation

$$\psi'' + \psi = F - \sigma M, \quad (\omega = 0) \tag{3.5}$$

This equation is obtained from Eqs (3.1) written as follows (T is the tangential component of the reaction): $J_1 \psi'' = -aT - b^2k_1 - bF$, $M - rT = 0$ after eliminating the unknown T .

By (3.5), the phase trajectories in the domain of sliding motions $\psi = -M$, $\omega = 0$ form a family of concentric circles

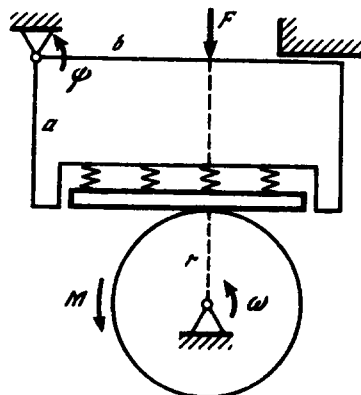


Fig. 5.

$$(\dot{\psi}')^2 + (\psi + F + \sigma M)^2 = \text{const} \tag{3.6}$$

about the point $\psi = -F - \sigma M, \psi' = 0, \omega = 0$.

The shape of the phase trajectories in the half-space $\omega > 0$ is determined by the relationship of the essential parameters f, σ and M , which agree in meaning with the corresponding parameters of the previous model. We note that the representative point, while in the half-plane $\omega \geq 0$ and moving in the domain $\psi \geq -M$, moves away from the boundary $\omega = 0$, but approaches it when moving in the domain $\psi \leq -M$. Hence it follows that a closed trajectory, if such exists, must contain a section of "sliding" motion, and so the search for a limit cycle [7] reduces to studying a point mapping of the half-line $\psi = -M, \psi > 0$ into itself (see Fig. 6, where the domain of "sliding" motions is hatched). The fact that ψ and ψ' vary independently of the variable ω makes it possible to consider the phase trajectories projected onto the (ψ, ψ') plane, and this considerably simplifies the investigation.

It follows from Eqs (3.3) and (3.5) that the qualitative division of the phase half-plane $\psi, \psi', \omega \geq 0$ into trajectories, projected onto the (ψ, ψ') plane, is determined by the mutual positions of the points $\psi_1 = F(\sigma - 1)^{-1}, \psi_2 = -M, \psi_* = -(F + \sigma M)$, and moreover

$$\psi_2 - \psi_* = \sigma\lambda(\sigma - 1)^{-1}, \quad \psi_1 - \psi_2 = \lambda(\sigma - 1)^{-1}, \quad \lambda = F + (\sigma - 1)M \tag{3.7}$$

Accordingly, we obtain bifurcation boundaries $F = 0, \lambda = 0$ and $\sigma = 1$. This bifurcation portrait is identical with the bifurcation portrait shown in Fig. 3 provided the domains C and D are combined, as are the domains E and F (the boundary $\lambda + 2M = 0$ turns out to be unimportant for the phase portrait).

Using (3.7) and Eqs (3.3), we find all possible qualitatively distinct divisions of the (ψ, ψ') plane into trajectories (Fig. 7). For values $\psi < 0, \psi' > 0$ only those trajectories on which the representative point starts its motion from the $\psi = -M$ plane are shown. Arrows indicate trajectories of "sliding" motions, in which allowance has been made for a small viscous friction converting the "centre"-type singular point into a singular point of the "stable focus" type. Curves with arrows to the right of the curve $\psi = 0$ represent an instantaneous jump of the representative point, due to the shoe impacting on the stop. As already remarked, the motion of the phase point ψ and ψ' is independent of ω . The variation of ω , for its part, is determined by the motion of the point (ψ, ψ') : it decreases when $\psi < -M$ and increases when $\psi > -M$.

According to Fig. 7, for all possible shapes of the phase portrait, a stable equilibrium may be reached after transients, either at the point $\psi = \psi' = 0$ or on the surface of "sliding" motions at the point $\psi = \psi_* < -M, \psi' = 0$. In the first case ω increases without limit (uniformly accelerated motion), in the second, $\omega = 0$. However, apart from such final motions, self-excited oscillations with respect to the variables ψ and ψ' may also occur, accompanied by periodic changes in ω with a positive mean.

Self-excited oscillations may be observed by constructing and investigating a point mapping of the half-line $\psi = -M, \psi' \geq 0$, since a closed trajectory may be formed in the ψ, ψ' plane only when that half-line is crossed. For example, in case A it follows from Fig. 6 that the point mapping $\bar{u} = \bar{u}(u)$, ($u = \psi'$) for trajectories in the $\psi = 0$ plane will consist of two transformations; T_0 and T_* , i.e. $T^{(1)} = T_*T_0$. Otherwise it will consist of T_1, T_2, T_3 and T_* , that is, $T^{(2)} = T_*T_3T_2T_1$.

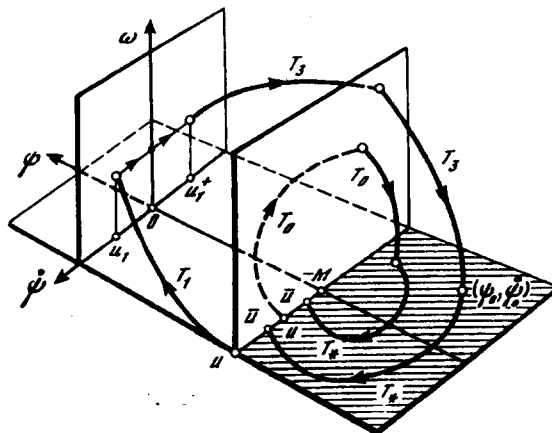


Fig. 6.

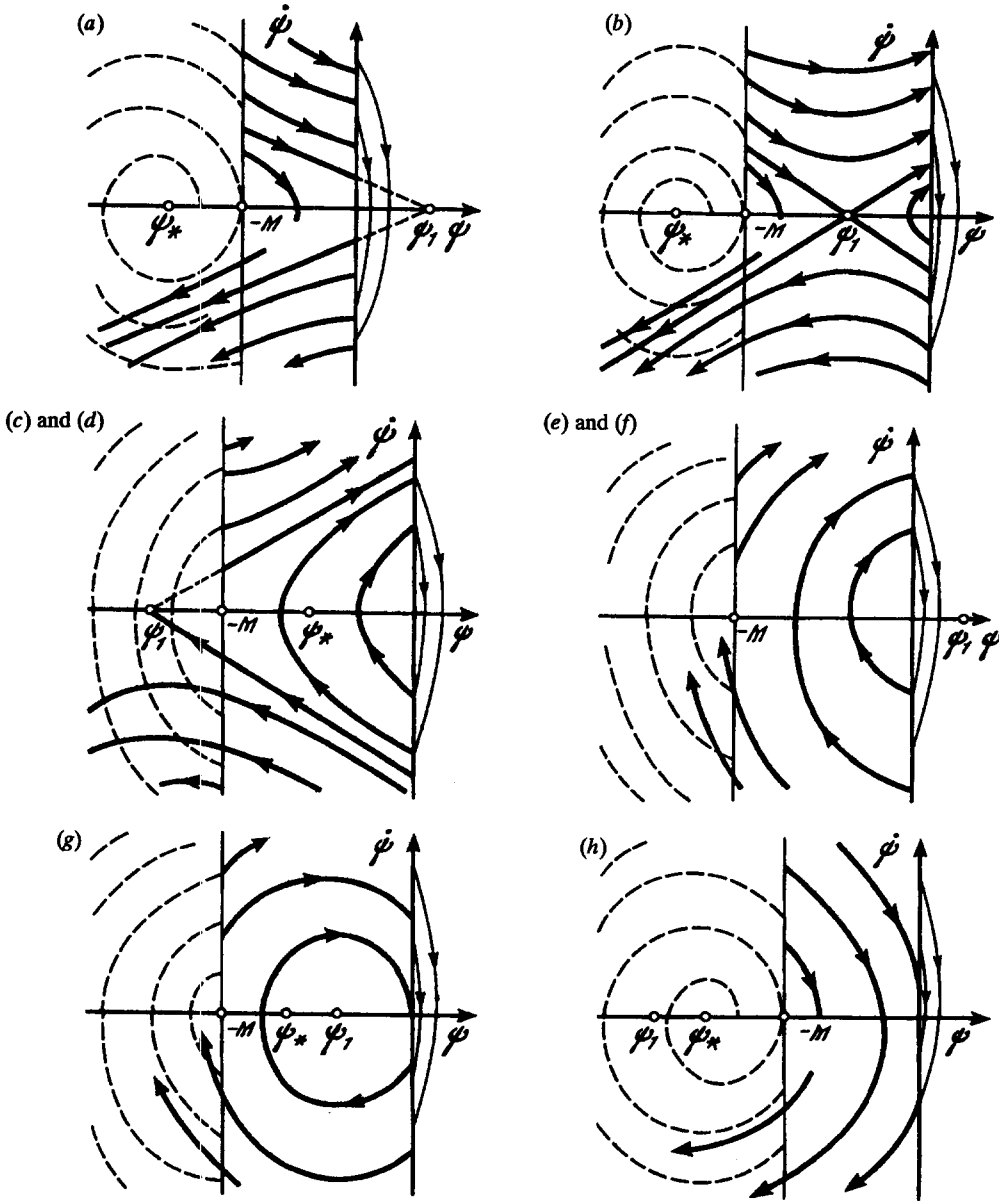


Fig. 7.

Let us first consider the case $T^{(1)} = T \cdot T_0$. In the domain A we have $\alpha^2 = \sigma - 1 > 0$, and so we can write Eqs (2.3) for $\psi < 0$ in the form

$$\psi'' - \alpha^2 \psi = -\alpha^2 \psi_1, \quad J\omega' = \psi + M \tag{3.8}$$

A solution of this equation for $\psi < 0$, satisfying initial conditions $t = 0, \psi = -M, \psi' = u, \omega = 0$, is

$$\begin{aligned} \psi &= \alpha^{-1} u \operatorname{sh} \alpha t + (M + \psi_1) \operatorname{ch} \alpha t + \psi_1, \quad \psi' = u \operatorname{ch} \alpha t - \alpha(M + \psi_1) \operatorname{sh} \alpha t \\ \alpha^2 J\omega &= u(\operatorname{ch} \alpha t - 1) - \alpha(M + \psi_1)(\operatorname{sh} \alpha t - \alpha t) \end{aligned} \tag{3.9}$$

Suppose that at a time $t = \tau_0 \alpha^{-1}$: $\psi = \psi_0, \psi' = \psi'_0, \omega = 0$. Then we proceed from the solution (3.9) to the relationship

$$\begin{aligned} \alpha \psi_0 &= u \operatorname{sh} \tau_0 - \lambda \alpha^{-1} \operatorname{ch} \tau_0 + F \alpha^{-1}, \quad \psi_0 = u \operatorname{ch} \tau_0 - \lambda \alpha^{-1} \operatorname{sh} \tau_0 \\ \alpha u &= (\operatorname{ch} \tau_0 - 1) - \lambda (\operatorname{sh} \tau_0 - \tau_0) = 0, \quad \lambda = F + (\sigma - 1)M, \quad \alpha^2 = \sigma - 1 \end{aligned} \tag{3.10}$$

These relations constitute the point mapping T_0 . To obtain the point mapping T^* , we consider the differential equation (3.5) and its integral (3.6), which can be written in this case as

$$(\psi')^2 + (\psi + \lambda + M)^2 = (\psi_0')^2 + (\psi_0 + \lambda + M)^2 \tag{3.11}$$

The representative point reaches the half-line $\psi = -M$, $\psi' > 0$ with coordinates $\psi = -M$, $\psi = u^-$. Substituting this into (3.7), we obtain

$$\bar{u}^2 = (\psi')^2 + (\psi_0 + \lambda + M) - \lambda^2 \tag{3.12}$$

Formulae (3.9) and (3.12) constitute the point mapping $T^{(1)} = T \cdot T_1$ for $u \leq u_*$, where the number $u_* = \sqrt{M(\lambda + F)}$ is determined from the condition that the phase trajectory must touch the straight line $\psi = 0$ at the origin of the (ψ, ψ') plane.

We will now determine an explicit expression for the point mapping $T^{(1)}$ for $u \ll 1$. In this case we deduce from equations (3.10) that

$$\psi_0 = -M - 3u^2(2\lambda)^{-1}, \quad \psi_0' = -2u$$

Substituting these expressions into (3.12), we get

$$\bar{u}^2 = u^2 + \frac{9}{4}\lambda^{-2}u^4 \tag{3.13}$$

Hence it follows that in the neighbourhood of the origin the Lamerey curve has the form of curve 1 in Fig. 8, indicating the possibility of oscillations with increasing amplitude. If viscous friction is taken into account, the Lamerey curve has the form of curve 2 in the neighbourhood of the origin. The point $u = \bar{u}$ corresponds to the existence of an unstable limit cycle in the phase space. Under perturbations $u < \bar{u}$ the system goes into the rest state, but under perturbations $u > \bar{u}$ oscillations occur, i.e. we have hard excitation of oscillations.

In order to show that self-excited oscillations exist in the system, it is sufficient to show that when $u > \bar{u}$ the Lamerey curve must cut the bisector of the coordinate angle in the reverse direction. This point may correspond to the existence in the phase space of a stable limit cycle, but in the unstable case it may indicate the existence of more complicated self-excited oscillations.

In terms of the notation of Fig. 6, when $u > u_*$ the point mapping $T^{(2)}$ consists of the sequence of transformations T_1, T_2, T_3 and T^* . Let us consider them in order. The transformation T_1 is obtained from the solution (3.9) of Eqs (3.8) with initial conditions $t = 0, \psi = -M, \psi' = u$ and $\omega = 0$. Suppose that at time $t = \tau_1 \alpha^{-1}$: $\psi = 0, \psi' = u_1$ and $\omega = \omega_1$. Then T_1 is determined by the expressions

$$\begin{aligned} \alpha u \operatorname{sh} \tau_1 - \lambda \operatorname{ch} \tau_1 + F &= 0, & \alpha u_1 &= \alpha u \operatorname{ch} \tau_1 - \lambda \operatorname{sh} \tau_1 \\ \alpha^3 J \omega_1 &= \alpha u (\operatorname{ch} \tau_1 - 1) - \lambda (\operatorname{sh} \tau_1 - \tau_1) \end{aligned} \tag{3.14}$$

The transformation T_2 , which describes the result of the inelastic impact of the stop, leaves the quantities

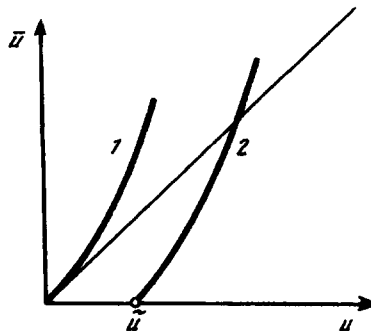


Fig. 8.

$\psi = 0$ and $\omega = \omega_1$ unchanged, but ψ' instantaneously takes the value $\psi'_+ = \kappa u_1$. To obtain T_3 , we use a solution of Eqs (3.8) satisfying the initial conditions $t = 0, \psi = 0, \psi' = \kappa u_1, \omega = \omega_1$.

Suppose that at time $t = \tau_2 \alpha^{-1}$ the phase variables take values $\psi = \psi_0, \psi' = \psi'_0, \omega = 0$. Then we find the following expression for T_3

$$\begin{aligned} \psi_0 &= -\kappa \alpha^{-1} u_1 \operatorname{sh} \tau_2 - \psi_1 (\operatorname{ch} \tau_2 - 1), & \psi'_0 &= -\alpha \psi_1 \operatorname{sh} \tau_2 - \kappa u_1 \operatorname{ch} \tau_2, \\ \alpha^2 J \omega_1 &= \alpha \psi_1 \operatorname{sh} \tau_2 + \kappa u_1 (\operatorname{ch} \tau_2 - 1) - \alpha (\psi_1 + M) \tau_2 \end{aligned} \tag{3.15}$$

Formulae (3.14) and (3.15), together with (3.12), represent the required point mapping $T^{(2)}$ for $u \geq u_* = \sqrt{M(\lambda + F)}$.

We will now find an explicit expression for the mapping $\bar{u} = \bar{u}(u)$ when $u \gg 1$. It follows from (3.18) that as $u \rightarrow \infty$ we have $\tau_1 \rightarrow 0, u \rightarrow \alpha M \tau_1^{-1}, u_1 \rightarrow \alpha M \tau_1^{-1}, J \omega_1 \rightarrow M \tau_1 (2\alpha)^{-1}$. Substituting this into (3.15), we obtain the following limiting relation

$$\frac{1}{2} \alpha M \tau_1 = \alpha \psi_1 \operatorname{sh} \tau_2 + \alpha \kappa M \tau_1^{-1} (\operatorname{ch} \tau_2 - 1) - \alpha (\psi_1 + M) \tau_2$$

which may hold as $\tau_1 \rightarrow 0$ if τ_2 also tends to zero. For $\tau_2 \ll 1$ this relation becomes a quadratic equation in τ_2 , the solution of which gives

$$\tau_2 = \tau_1 \kappa^{-1} (1 + \sqrt{1 + \kappa})$$

Hence, using (3.15), we obtain

$$\lim_{u \rightarrow \infty} \psi_0 = -M(1 + \sqrt{1 + \kappa}), \quad \lim_{u \rightarrow \infty} \psi'_0 = -\kappa u$$

Substituting these expressions into (3.12), we obtain

$$\bar{u}^2 = \kappa^2 u^2 + M^2(1 + \kappa) - 2\lambda M \sqrt{1 + \kappa}$$

Thus, as u increases, the curve $\bar{u} = \bar{u}(u)$ approaches an asymptote $\bar{u} = \kappa u + \text{const}$ ($\kappa < 1$), which certainly cuts the bisector of the first quadrant in the (u, \bar{u}) plane. This point of intersection indicates that the phase space of the system contains either a stable limit cycle or more complicated self-excited oscillations.

Computers have been used to construct the point mapping $\bar{u} = \bar{u}(u)$, using formulae (3.10)–(3.12) for $\bar{u} < \bar{u}_*$ and formulae (3.13)–(3.15) for $u > u_*$, with the following numerical values of the physical parameters in the domain A (Fig. 3): $\sigma = 1.36, M = 1.0, F = 0.36, \kappa = 0.44$ and $u_* = 1.04$. The resulting

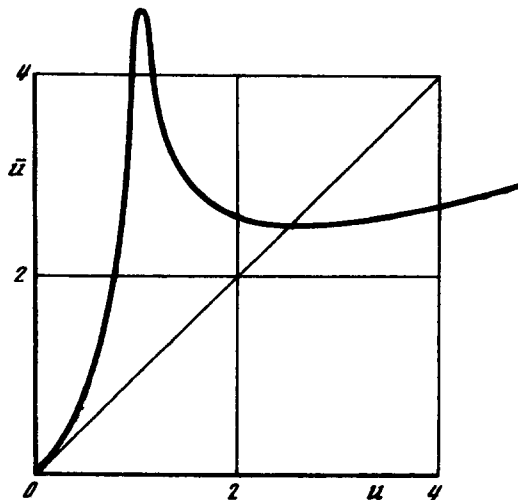


Fig. 9.

Lamerey diagram is shown in Fig. 9. It corresponds to the existence of simple, single self-excited oscillations. At the point where the Lamerey diagram cuts the bisector, the derivative with respect to the modulus is less than unity.

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